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THE INFLUENCE OF POROSITY ON THE ELASTIC RESPONSE OF HOMOGENEOUS AND STRUCTURAL COMPOSITE MEDIA

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Abstract—Continuum theories of composites are employed to analyze the influence of inclusions and porosity on the elastic response of both homogeneous and laminated composite media. The general model analyzed consists of a periodic array of two perfectly bonded laminates; one of which consists of an elastic homogeneous material while the other is made up of a periodic array of cylindrical elastic inclusions that are distributed in another elastic matrix material. Several specific models are deduced as special cases. In all cases, porosity is simulated in the limit as the properties of the inclusions identically vanish. It is demonstrated that porosity plays a major role in the geometric dispersion of such media; in particular, it increases the arrival and rise times (spreading) of a propagating transient pulse. For the special case of elastic inclusions in a homogeneous matrix media, the present results correlate very well with existing experimental data and other approximate analyses.

INTRODUCTION

In recent years considerable effort has been expended upon the modeling, testing, and analysis of structural composites. For most realistic composites, however, the variability in constituent properties, geometrical arrangements, and mechanical and thermal interactions render exact analytic investigations impossible. As an alternative, investigators have sought effective analytic models which, in general, are appropriate for periodically inhomogeneous linear materials. Some of these models have been complemented with numerical code calculations to account for other physical effects such as the material's dispersivity, nonlinearity, and fracture. In most cases the combined efforts of the analytical and numerical investigations have predicted and correlated well with experimental results. A current survey of stress wave propagation in composite materials is given in [1].

As pointed out by Peck[1], a particular item, which has not been considered so far but could be important, is the geometric dispersion caused by porosity. In fact, experiments performed on a specific composite (three dimensional quartz phenolic) in the low pressure regime indicate that porosity is capable of exerting considerable influence on geometric dispersion[2]. Generally speaking, the term porosity indicates the existence of voids distributed in a particular material. As for structural composites, micrographs indicate the voids are often somewhat periodic and frequently possess a particular geometry. These trends are perhaps due to the periodicity inherent in structural composites and to the nature of their manufacturing processes.

While little is known about the influence of porosity on the response of structural composites, experimental and analytical investigations of such influence on homogeneous porous materials are fairly well developed, particularly with regard to the use of such materials for shock wave attenuation. The most widely used model (known as the $P-\alpha$ model) has been suggested by Herrmann[3] for porous ductile materials in the high pressure, nonlinear regime. In the low pressure, elastic regime, little data are available, and different models lead to different estimates of the effective sound speed. The $P-\alpha$ model, for example, leads to the unrealistic conclusion that the elastic wave speed could be higher than the speed in the solid material. Actually, the

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wave speed in a porous material is lower than that in the corresponding solid. This observation has been asserted by Carroll and Holt[4] who modified the $P - \alpha$ model accordingly. Support for Carroll and Holt's assertion may be found in the articles by Sve[5] and Kirsch[6] with results deduced directly from the theory of elasticity.

The evidence indicates that more accurate predictions of the response of composites can be made if the effects of the presence of voids are taken into account. It is anticipated that the size, geometry and distribution of the voids can all play a significant role in the elastic response of the material. In this paper we study a simple idealized porous model in order to demonstrate the qualitative influence of porosity on the propagation of stress waves in homogeneous and composite elastic materials, particularly with regard to the distrubance acoustic speed, rise time, and attenuation.

COMPOSITE MODELING

The specific composite which we will analyze is depicted in Fig. 1. It consists of a periodic array of two laminates with perfect bonds at their interfaces. It is assumed that one of the laminates is composed of a linear elastic homogeneous material while the other is made up of a periodic array of cylindrical holes that are distributed in another elastic matrix material. The choice of the cylindrical shape for the voids is made in light of its mathematical tractability and the ease in which it can be used to investigat the influence of void geometry on the response.

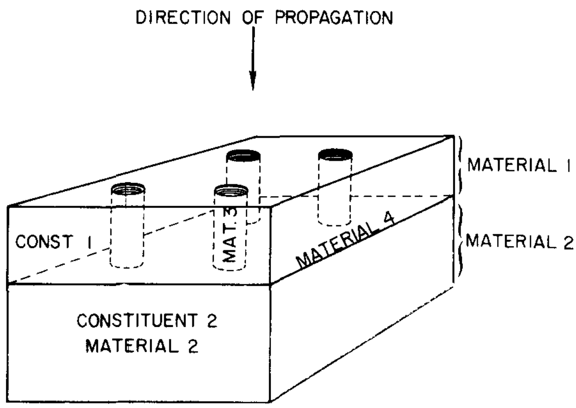

Fig. 1. Composite's geometry.

In the following we shall consider wave propagating normal to the laminates (i.e. parallel to the voids). With regard to the mathematical analysis, we follow the following steps: (a) We first assume that the voids are filled with an elastic homogeneous isotropic media; (b) subsequently, we replace the composite with a homogeneous, but dispersive, higher order continuum model. This is done by firstly homogenizing the composite laminate via a static consideration of the continuum mixture theory of Hegemier, Gurtman and Nayfeh[7] and subsequently combining it with the adjacent homogeneous laminate as per the asymptotic analysis technique developed by Hegemier and Nayfeh[8] to treat the laminated plates. The basic physical assumption made in this homogenizing procedure is that the wave length of the particular disturbance being propagated is long with respect to the microstructure of the composite; (c) finally, the influence of the voids will be simulated in the limit as the properties of the material filling them vanish identically. Support for this simulation may be inferred from works by Mackenzie[9] and

Hashin[10]. Mackenzie determined effective elastic moduli for a slightly porous material containing small spherical holes. Hashin obtained bounds for the moduli of an elastic medium containing finite concentration of spherical elastic inclusions. To the order of approximation in porosity, Hashin's results reduce exactly to those of Mackenzie in the limit as the material properties of the elastic inclusions vanish.

To avoid possible notational complexity, we shall refer to the "composite laminate" as constituent 1, to its fiber and matrix components as material 3 and 4, respectively, and to the "homogeneous laminate" (material 2) as constituent 2.

ANALYSIS

Homogenization of the composite laminate

For "dilatational"-type waves propagating parallel to the filled voids, the composite laminate can be modeled as concentric cylinders, rigidly bonded at their interfaces and subject to vanishing shear stress and radial displacement on their outer boundary. For this modeling the static effective mass density and modulus for constituent 1 are given in [8] as

$$\rho_1 = \rho_3 n_3 + \rho_4 n_4, \tag{1a}$$

$$E_1^* = E_3 n_3 + E_4 n_4 - (\lambda_3 - \lambda_4)^2 / D, \tag{1b}$$

$$D = \frac{1}{n_3 n_4}[(\lambda_3 + \mu_3)n_4 + (\lambda_4 + \mu_4)n_3 + \mu_4], \tag{1c}$$

where ρ_β and n_β, $\beta = 3, 4$, denote the actual mass-density and volume fraction of material β. Here λ_β, μ_β and $E_\beta (= \lambda_\beta + 2\mu_\beta)$ are the Lamé elastic constants and stiffness of material β. For future reference, we now define the wave speeds in the actual materials 2, 3, 4 as

$$c_j = (E_j / \rho_j)^{1/2}, \qquad j = 2, 3, 4. \tag{2}$$

From (1a, b) we construct a representative wave speed as

$$c_1 = (E_1^* / \rho_1)^{1/2}. \tag{3}$$

Special cases of (3) in which one of the constituents material properties identically vanish can be constructed. These result in known expressions which may lend support to the presently proposed idea of simulating voids in the limit of vanishing material properties. For example, if we set $\rho_4 = \mu_4 = \lambda_4 = 0$, then (3) yields

$$c_{1r} = [\mu_3(3\lambda_3 + 2\mu_3)/\rho_3(\lambda_3 + \mu_3)]^{1/2} \equiv (Y_3/\rho_3)^{1/2}, \tag{4}$$

where Y_3 is the Young's modulus of material 3. The result (4) represents the well-known acoustic wave speed (designated c_{1r}) of a solid circular rod. As another example whose results will be used later, if we set $\rho_3 = \mu_3 = \lambda_3 = 0$, then (3) yields

$$c_{1h} = \left[\frac{1}{\rho_4}\left\{E_4 - \frac{\lambda_4^2}{E_4 + \mu_4(n_4/n_3)}\right\}\right]^{1/2}, \tag{5}$$

which appears to define the wave speed, c_{1h}, for an elastic infinite space containing infinite cylindrical "holes" distributed in a periodic fashion. Notice now that the surfaces of the holes are stress free. In the limit as $n_4 \to 0$, (corresponding to thin walled cylinders), (5) further reduces to

$$c_{1p} = [4\mu_4(\lambda_4 + \mu_4)/\rho_4 E_4]^{1/2} \equiv [Y_4/\rho_4(1 - \nu_4^2)]^{1/2} \tag{6}$$

which is the known extensional wave speed, c_{1p}, of a rectangular plate. The result in (6) is expected; it can be easily explained if one imagines unwrapping every two adjacent thin walled cylinders to form a rectangular plate. The outer boundaries of the cylinders coincide and form the midplane of the plate and their inner boundaries define its two outer (stress free) faces. Notice also that the vanishing of the shear stress and transverse displacement at the outer surfaces of the cylinders implies vanishing of these quantities at the midplane of the plate. These symmetry and boundary conditions are consistent with the plate theory analysis.†

Homogenization of the total composite

With the representative material properties (1a) and (1b) of the composite laminate being defined, the present composite reduces to that of a laminated plates model. Consequently, the technique developed by Hegemier and Nayfeh[8] to treat the laminated plates problem becomes directly applicable. Using the analysis and results of [8], we replace the total composite by a homogeneous, but dispersive, higher order continua whose so-called "far field" motion is described by the global fourth order partial differential equation (see [8], equation 61 in dimensional form),

$$[c^2\partial x^2 - \partial_t^2 + G\partial_t^4][\Phi] = 0. \tag{7}$$

Here Φ represents either stress or displacement of constituent 1 or 2, c and G define the composite's zero frequency limit wave speed and the composite's dispersion parameter, respectively. In terms of the material properties and the geometric arrangements of the constituents 1 and 2, the values of c and G are given by

$$c = (E_c/\rho_c)^{1/2}, \tag{8}$$

$$G = \frac{\Delta^2 c^2 n_1^2 n_2^2}{3}\left[\frac{\rho_1}{E_2} - \frac{\rho_2}{E_1^*}\right]^2, \tag{9}$$

where

$$E_c = E_1^* E_2/(E_1^* n_2 + E_2 n_1), \tag{10a}$$

$$\rho_c = \rho_1 n_1 + \rho_2 n_2. \tag{10b}$$

Here ρ_2 is the actual mass density of material 2 (constitutent 2); E_1^* and ρ_1 are the corresponding effective properties of constituent 1 as defined in equations (1) and (2); n_α, $\alpha = 1, 2$, define the volume fractions of the constituents α, i.e.

$$n_\alpha = h_\alpha/\Delta, \qquad \Delta = h_1 + h_2, \tag{11}$$

where h_α is the half-thickness of the laminate α.

†To further strengthen the idea of simulating voids by vanishing material properties, one can easily show, starting from Rytov's[11] exact dispersion relations for the laminated wave guides, that the plate's exact dispersion relation[12] can be obtained if the material properties of one of the wave guides' constituents vanish identically.

The dependence of c and G on the material properties and the geometric arrangements of the composites constituents are of great importance particularly with regard to the manner in which they influence the arrival time and the distortion (spreading) shape of the propagating pulse in the medium. This influence is best demonstrated by examining the low frequency (parabolic approximation) dispersion relation of equation (7)

$$c_\omega = c\left[1-\left(\frac{G\omega^2}{2}\right)\right], \tag{12}$$

and the corresponding well-known head-of-the-pulse (see, e.g. [13]) solution for the propagation of a unit boundary input; namely,

$$\Phi(x,t) = \frac{1}{3} - \int_0^\xi Ai(\eta)\,\mathrm{d}\eta, \tag{13}$$

where

$$\xi = \left(t - \frac{x}{c}\right)\Big/\tau, \qquad \tau = (3Gx/2c)^{1/3}. \tag{14}$$

Here Ai is the Airy function, ξ is a nondimensional time and τ is called the characteristic dispersion time which depends, for a given propagation distance x, on the constituent's material properties and geometric arrangements through c and G. The behavior of $\Phi(x, t)$ as a function of t is depicted in Fig. 3. The pulse begins with a steady rise, followed by oscillations about the boundary input. The wave is roughly a step arriving at $\xi = 0$, so that the main disturbance is propagating at the composite's speed c. The disturbance rise time, i.e., the time that the disturbance takes to rise from 0^+ to its first peak, is proportional to τ.

Equations (8) and (9), as they stand, define the effective wave speed and dispersion parameter for a laminated composite which also includes cylindrical elastic inclusions (material 3). In the following we shall closely investigate some special cases of the above results in order to demonstrate the applicability of the present analysis to a wide variety of physically and industrially relevant models.

Hologeneous materials with inclusions

For the special case where material 2 is the same as material 4 these equations will then define the wave speed and dispersion parameter for a homogeneous elastic medium including a periodic array of cylindrical elastic inclusions. The results of this special case, especially those pertaining to the effective static properties, make it possible to check on the accuracy of our analysis by comparing them with the available experimental data and other analyses reported by Paul [14]. In his paper Paul developed an expression to give an approximate effective Young's modulus of a matrix material containing cubical inclusions which, for the sake of comparison, can be written in the present notation as

$$\frac{Y_c}{Y_2} = \frac{Y_2 + (Y_3 - Y_2)g^{2/3}}{Y_2 + (Y_3 - Y_2)(1 - g^{1/3})g^{2/3}}, \tag{15}$$

where g is the volume fraction of the cubical inclusions. Paul also compared his results, as shown in Fig. 2, with the experimental data reported by Nishimatsu and Gurland [15], and Kieffer and

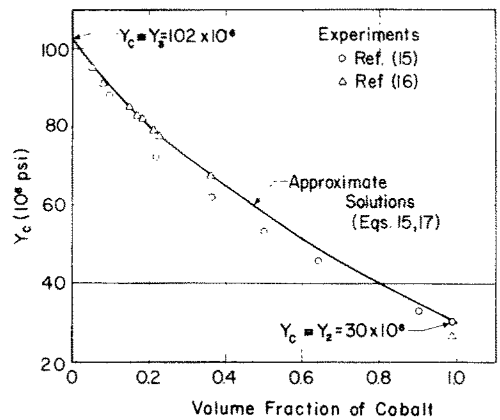

Fig. 2. Comparison of theory with experiments.

Schwartzkopf[16] for an alloy system of tungsten carbide and cobalt. In (17) the subscripts 2 and 3 refer to the cobalt (matrix) and tungsten (inclusion), respectively. The necessary properties used are given by

$$Y_2 = 30\times10^6\,\text{psi},\quad Y_3 = 102\times10^6\,\text{psi},\quad \nu_2 = 0{\cdot}3\quad\text{and}\quad \nu_3 = 0{\cdot}22.$$

Unfortunately the available experimental data concern the measurement of the composites effective Young's moduli, Y_c, rather than its effective stiffness, E_c. Thus in order to directly compare our results with the experimental results we need to derive a second effective property for the composite. One such property which, in light of our analytical procedure, can be easily derived is the shear modulus μ_c. Consistent with our derivation of E_C we construct an expression for μ_c as

$$\mu_c = \mu_1^*\mu_2/(\mu_1^*n_2 + \mu_2 n_1), \tag{16a}$$

where

$$\mu_1^* = \mu_3 n_3 + \mu_4 n_4. \tag{16b}$$

Using known relations, we construct an effective Y_c for the laminated composite with cylindrical inclusions from (10a) and (16a) as

$$Y_c = E_c - \frac{(E_c - 2\mu_c)^2}{(E_c - \mu_c)}. \tag{17}$$

The expression (17) can be directly compared with Paul's results after setting the properties of material 4 equal to those of material 2 and subsequently choosing the inclusion's cylindrical shape which closely approximates a cube. This can be done by setting $n_3 = n_1^2$ with $g = n_1 n_3 \equiv n_1^3$ defining the volume fraction of the inclusions.† Using the constituent's properties mentioned above, numerical evaluation of expression (17) and Paul's formula are found to correlate very well as shown in Table 1. In this table both results are normalized with respect to Y_2.

†Notice that this condition defines a class of infinitely many geometric arrangements of the cylindrical inclusions. This is because no restriction has been placed on the actual dimensions of the inclusions. One of these arrangements closely approximates a cube, however.

Table 1.

Inclusion volume fraction $g = n_1^3$	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
Paul's Formula (15)	1	1·1879	1·358	1·5312	1·7149	1·9147	2·1363	2·3860	2·6721	3·0052	3·4
Present Formula (17)	1	1·1897	1·3604	1·5339	1·7177	1·9177	2·1392	2·3888	2·6745	3·0068	3·4

Porous composites

If subsequently we substitute from equations (1) and (2) into (8), and let the properties of material 3 vanish, the wave speed in the porous composite, c_p, takes the form

$$c_p = \left\{\frac{E_2E_4}{[n_2(\rho_2-\rho_4)+\rho_4(1-p)][n_2E_4+n_1R_1E_2]}\right\}^{1/2}, \tag{18}$$

where p is the porosity defined as

$$p = n_1n_3, \tag{19a}$$

and

$$R_1 = \frac{1+[2n_3(1-\nu_4)/n_4(1-2\nu_4)]}{1+[n_3(1+\nu_4)/(1-\nu_4)]}, \tag{19b}$$

a function of the Poisson's ratio ν_4. In deriving (19b) we have used the relation $\nu = \lambda/2(\lambda+\mu)$. Similarly, the porous composite's characteristic dispersion time, τ_p, becomes

$$\tau_p = \left[\frac{\Delta^2n_1^2n_2^2xc_p}{2}\left\{\frac{\rho_1}{\rho_2c_2^2}-\frac{\rho_2}{\rho_1c_{1h}^2}\right\}^2\right]^{1/3}, \tag{20}$$

where c_{1h} is given in (5) which, in terms of ν_4, can also be written as

$$c_{1h}^2 = c_4^2R_2, \tag{21}$$

$$R_2 = 1-\frac{2\nu_4^2}{(1-\nu_4)^2\left\{2+\frac{n_4}{n_3}\left(\frac{1-2\nu_4}{1-\nu_4}\right)\right\}}. \tag{22}$$

Results for the nonporous composite can be obtained by setting $n_3 \equiv 0$ either in equations (14) and (20) or in (8) and (9). For the nonporous case, R_1 and R_2 become unity and consequently equations (14) and (20) reduce to the wave speed, c_0, and the dispersion parameter, τ_0, of the solid composite, namely

$$c_0 = \left[\frac{E_2E_4}{(\rho_2n_2+\rho_4n_1)[n_1E_2+n_2E_4]}\right]^{1/2}, \tag{23}$$

$$\tau_0 = \left[\frac{\Delta^2n_1^2n_2^2xc_0}{2}\left\{\frac{\rho_1}{\rho_2c_2^2}-\frac{\rho_2}{\rho_1c_4^2}\right\}^{1/2}\right]^{1/3}. \tag{24}$$

On the other hand, as $n_3 \to 1$, the porosity p approaches the value n_1, $R_1 \to \infty$, $R_2 \to (1-2\nu_4)/(1-\nu_4)^2$, and consequently $c_p \to 0$ and $\tau_p \to \infty$. These latter limiting results correspond to a model of porous material consisting of a series of identical plates of solid material separated by gaps. For this plate-gap model, it is obvious that a wave propagating normal to the plates cannot be transmitted.

In order to demonstrate the influence of porosity on the propagation speed, arrival time and characteristic dispersion time of the composite, we utilize the following material properties and volume fractions:

$$\rho_2 = 5{\cdot}0\ \text{g/cm}^3, \qquad \rho_4 = 1{\cdot}0\ \text{g/cm}^3,$$
$$E_2 = 1\times 10^{13}\ \text{dynes/cm}^2, \quad E_4 = 1\times 10^{11}\ \text{dynes/cm}^2,$$
$$n_1 = 0{\cdot}4, \qquad \nu_4 = 0{\cdot}3.$$

With these properties the values of nonprous wave speed, c_0, is calculated as $c_0 = 2{\cdot}6915\times 10^5$ cm/sec. With this value of c_0, and for the sake of simplicity in the numerical comparison, we choose $x = 0{\cdot}26915$ cm which yields the arrival time x/c_0 as $1{\cdot}0\times 10^{-6}$ sec. We also choose $\Delta = 0{\cdot}13874$ cm such that the characteristic dispersion time of the nonporous composite, τ_0, as calculated from (21b) is normalized to 0·1 sec. With these referral values of c_0 and τ_0, and fixing $n_1 = 0{\cdot}4$, $\nu_4 = 0{\cdot}3$, Fig. 3 demonstrates the influence of porosity on the arrival time and spreading of the propagating step pulse for the given monitoring station $x = 0{\cdot}26915$ cm. This figure demonstrates the important phenomenon that porosity tends to increase both the arrival time and spreading of the pulse.

For the unit step pulse, we see from Fig. 3 that dispersion induces an overshoot in the composite's response regardless of the porosity. For finite-width pulses, however, dispersion will often cause the initial disturbance to attenuate; the amount of attenuation will depend very much on the percent porosity and to a greater extent on the void shape. With reference to Fig. 3, it is a straightforward matter to demonstrate this conclusion simply by constructing the response of the porous composite to a rectangular pulse of duration τ_0. This can be done by taking the difference of two separate step loads applied at times τ_0 apart.

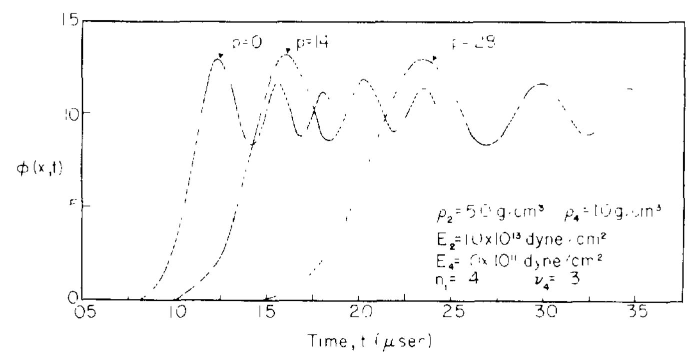

Fig. 3. Far-field wave profiles as functions of actual time for different porosities.

Homogeneous porous materials

For homogeneous porous materials, we set in (18–22) the properties of material 2 equal to those of material 4. For this special case c_p further reduces to

$$c_{hp} = c_4\left[\frac{1}{(1-p)(n_2+n_1R_1)}\right]^{1/2} \tag{25}$$

which in turn modifies the characteristic dispersion time in (20). The nonvanishing of this resulting dispersion time indicates that homogeneous porous materials are also dispersive. This dispersion is perhaps due to the fact that the voids are assumed to be distributed in a periodic manner. Actual homogeneous porous materials are, however, less dispersive due to the inherent random distribution of the voids. For the special case of cubical holes, i.e., $p = n_1n_3$, and $n_3 = n_1^2$, the variation of the wave speed with porosity as given in (25) can be compared with the corresponding result that can be extracted from the formula (38) reported in [5]. The difference between the two models, of course, is that the cylindrical periodic voids treated here are redistributed as spherical voids in [5] while maintaining all other parameters the same. In our present notation, formula (38) in [5] takes the simple form

$$c_{hp} = c_4\left[\frac{g_0}{(1-p)(n_1+g_0n_2)}\right]^{1/2}, \tag{26a}$$

with

$$g_0 = \frac{1-2\nu_4}{1-2\nu_4+[(1+\nu_4)p_1/2]} - \frac{10p_1(1-2\nu)}{7-5\nu_4+2(4+5\nu_4)p_1}, \tag{26b}$$

and

$$p_1 = p/n_1, \tag{26c}$$

where p is the porosity in the total medium.

Using the properties of the solid material (presently referred to as material 4), $\rho_4 = 1{\cdot}0\ \mathrm{g/cm^3}$, $E_4 = 10^{10}\ \mathrm{dynes/cm^2}$. The variation of the wave speeds (25) and (26) as functions of porosity for different values of ν_4 are depicted and compared in Fig. 4. These results are found to bracket the corresponding variation of the speed for the case where the voids are distributed randomly everywhere in the material, namely for $n_1 = 1$ in equation (26).

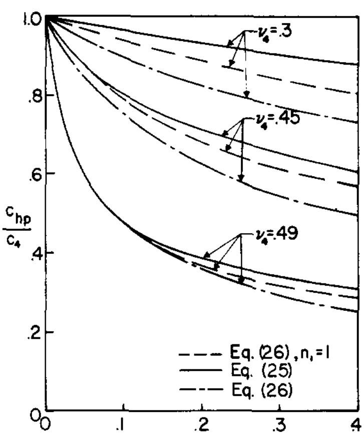

Fig. 4. Variations of wave speeds as functions of porosity for different Poisson's ratios.

Finally, in Fig. 5 we demonstrate the influence of the variation of the porous layer volume fraction, n_1, on the propagation speed for a given percent porosity. For the given fixed porosity $p = 0{\cdot}1$, the solid lines illustrate the variation of the normalized speeds, c_{hp}/c_4, as functions of n_1. The two solid lines are drawn for the two Poisson's ratios 0 and 0·45 as also labeled on the figure. For the sake of comparison similar results are illustrated for the fixed percent porosity $p = 0{\cdot}3$. As can be seen from this figure, the distribution of voids has profound influence on the material response especially in cases when n_1 is small (i.e., $n_3 \to 1$) which indicates that speed will drop very rapidly when the wave encounters a large void; it reduces to zero in the limit of the "plate-gap" model mentioned earlier. Also shown in this figure (broken-dotted lines) are the corresponding results obtained from equation (26).

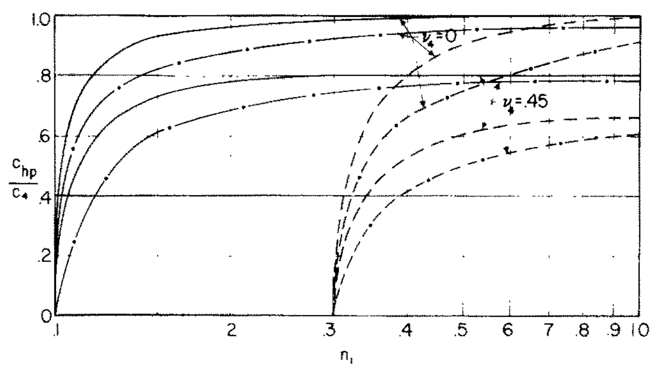

Fig. 5. Variations of wave speeds as functions of void's shapes for given porosities and Poisson's ratios.

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